A certified wavelet-based physics-informed neural network for nonlinear model reduction of parameterized partial differential equations

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Parameterized partial differential equations (PPDEs)

$$B_\mu : X \to Y', f_\mu \in Y', \mu \in \mathcal{P} \subset \mathbb{R}^p : \quad B_\mu u_\mu = f_\mu \tag{1}$$

arise to describe physical phenomena. These equations often have to be solved either in a multi-query or in a real-time context for different parameters μ resulting in the need for model order reduction. For linear, coercive and affine elliptic as well as parabolic PPDEs it is known that linear projection-based methods, e.g. the reduced basis method [7] are working well. Whereas, for transport- or wave-type problems it has been proven that the Kolmogorov N-width decay is poor ([6], [4]), such that linear model reduction techniques are bound to fail and nonlinear methods are needed.

The recent success in solving various PDEs with neural networks (NNs), particularly with physics-informed NNs (PINNs) (see e.g. [8] [1]) suggests that they are a natural candidate for nonlinear model order reduction (MOR) techniques, although an a-posteriori error control is lacking. Usually the solution of a PPDE is unknown, such that the training of a physics-informed NN exploits the strong-form residual of (1) as a loss function and a penalty term is added to enforce boundary and/or initial conditions. With that at hand, a target function is defined to determine the parameters of the NN during an optimization phase. But to train a quantity which is an upper bound for the error, one has to rely on certain variational formulations of (1).

Our method faces the aforementioned problems for the parameterized transport and wave equation by using the ultra-weak variational formulation ([3], [2], [5]), which involves equality of error and residual. Therefore, the dual norm of the residual

$$\|r_\mu\|_{Y'} := \|f_\mu - B_\mu u_\mu\|_{Y'} \tag{2}$$

gives us not only a-posteriori information about the error of an approximation, but also a loss function to train the PINN. To evaluate the dual norm of (2) we expand the residual in a wavelet basis and exploit the norm equivalences of these. The target function to train the PINN is then defined as a sum of dual norms for a finite sample set $\mathcal{S} \subset \mathcal{P}$ of parameters μ. Due to the fact, that the boundary and initial conditions are encoded in the residual as a part of f_μ we do not need a penalty term. As a result of the training process, the PINN is a nonlinear approximation of the mapping $\mu \mapsto u_\mu$. The application of the method is shown in numerical experiments.

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