Probabilistic reduced basis method for parameter-dependent problems

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In this talk, we consider the approximation of some function $u : \xi \mapsto u(\xi)$, defined on $\Xi \subset \mathbb{R}^p$ and taking its values in some vector space V, by means of a Reduced Basis (RB) method. It aims at constructing an approximation $u_n(\xi)$ of $u(\xi)$ as a projection in a finite dimensional space $V_n \subset V$ generated by snapshots of u selected through a greedy procedure. A practical algorithm is as follows. Given $\{\xi_1, \dots, \xi_{n-1}\}$ and the corresponding subspace $V_{n-1} = \text{span}\{u(\xi_1), \dots, u(\xi_{n-1})\}$, a new parameter value ξ_n is selected as

$$\xi_n \in \arg\max_{\xi \in \tilde{\Xi}} \Delta(u_n(\xi), \xi), \tag{1}$$

with $\Delta(u_n(\xi), \xi)$ a suitable error estimate and $\tilde{\Xi} \subset \Xi$ a finite training set. Recently, probabilistic variants of this algorithm have been considered. In [4], different training sets $\tilde{\Xi} = \Xi_n$ randomly chosen are used at each step. In [2, 3], a control variate using a RB paradigm has been proposed where $\Delta(u_n(\xi), \xi)$ is a Monte Carlo estimate of the variance of the projection error. In the lines of [2, 3], we propose a probabilistic greedy algorithm in which $\Delta(u_n(\xi), \xi)$ is expressed as the expectation of a parameter-dependent random variable $Z_n(\xi)$, i.e.

$$\Delta(u_n(\xi), \xi) = \mathbb{E}(Z_n(\xi)). \tag{2}$$

To solve (1) with (2), a Probably Approximately Correct bandit algorithm [1] is retained. It returns a quasi-optimum in relative precision, with high probability. The resulting greedy algorithm is proven to be a weak greedy algorithm with high probability.

Applications concern the approximation of $u(\xi) : D \to \mathbb{R}$, $D \subset \mathbb{R}^d$, for which we only have access to pointwise evaluations at $(x, \xi) \in D \times \Xi$. For u a costly functional given a priori, the proposed method is applied to compute u_n an interpolant of u in V_n. In that case, it can be seen as a probabilistic empirical interpolation method. Then, we consider a probabilistic RB method for the solution of parameter-dependent partial differential equations on D. Here, as u is not explicitly given, the proposed method relies on pointwise estimates $u(x, \xi)$ obtained through the so-called *Feynman-Kac representation formula.*

References

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