A Differential Geometric Formulation for Model Order Reduction on Manifolds

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Differential geometric formulations are central in multiple classes of dynamical systems like e.g. Lagrangian systems. In classical Model Order Reduction (MOR) these underlying differential geometric formulations are typically not reflected. This becomes especially problematic for recent MOR approaches which use nonlinear techniques to approximate the state [1, 2, 3] (in the following referred to as MOR on manifolds) since inconsistent projections of the full-order model might be used, see [2, Remark 3.5 (Alternative Galerkin projection)]. In this work, we present how MOR (on manifolds) can be formulated for basic differential geometric objects like vector fields and covector fields. This rigorous formulation has the advantage that it clearly shows how to project the full-order system and thereby inherently prevents inconsistent projections. Sparked by the insights of the new differential geometric formulation, we discuss extensions to the training of nonlinear autoencoders $(\boldsymbol{d}(\cdot,\boldsymbol{\theta}), \boldsymbol{e}(\cdot,\boldsymbol{\theta}))$ based on a dataset $X := \{\boldsymbol{x}_i\}_{i=1}^{n_\mathrm{s}} \subset \mathbb{R}^N$

$$\begin{aligned} \boldsymbol{d}(\cdot,\boldsymbol{\theta}) &: \mathbb{R}^n \to \mathbb{R}^N, \\ \boldsymbol{e}(\cdot,\boldsymbol{\theta}) &: \mathbb{R}^N \to \mathbb{R}^n, \end{aligned} \qquad \text{choose } \boldsymbol{\theta} \in \mathbb{R}^{n_\mathrm{p}} \text{ such that} \qquad (\boldsymbol{d}(\cdot,\boldsymbol{\theta}) \circ \boldsymbol{e}(\cdot,\boldsymbol{\theta}))(\boldsymbol{x}_i) \approx \boldsymbol{x}_i \quad \text{for } 1 \le i \le n_\mathrm{s}.$$

These extensions include two ideas: (i) The modification for exact reproduction of initial values from [2, Section 5.3] is adopted in the formalism and respected in the training procedure. This guarantees that the same coordinates are used in the training and the evaluation of the autoencoder and results in a modified loss function

$$\mathcal{L}_\mathrm{D}(\boldsymbol{\theta}) := \sum_{i=1}^{n_\mathrm{s}} \left\| (\boldsymbol{d}(\cdot,\boldsymbol{\theta}) \circ \boldsymbol{e}(\cdot,\boldsymbol{\theta}))(\boldsymbol{x}_i) - (\boldsymbol{d}(\cdot,\boldsymbol{\theta}) \circ \boldsymbol{e}(\cdot,\boldsymbol{\theta}))(\boldsymbol{0}) - \boldsymbol{x}_i \right\|^2. \tag{1}$$

As a second extension, (ii) snapshots of the right-hand side $X_{\boldsymbol{f}} := \{\boldsymbol{f}_i\}_{i=1}^{n_\mathrm{s}}$ are included in the training by introducing an additional loss function

$$\mathcal{L}_{\boldsymbol{f}}(\boldsymbol{\theta}) := \sum_{i=1}^{n_\mathrm{s}} \left\| \left(\boldsymbol{I}_N - \mathrm{D}_{\boldsymbol{x}}(\boldsymbol{d}(\cdot,\boldsymbol{\theta}) \circ \boldsymbol{e}(\cdot,\boldsymbol{\theta}))\big|_{\boldsymbol{x}_i} \right) \boldsymbol{f}_i \right\|^2. \tag{2}$$

Appropriate balancing of the both losses, (1) and (2), is discussed. In the numerical experiment, we consider the reduction of the one-dimensional Burgers' equation via a so-called deep convolutional autoencoder from [2] and investigate how the suggested extensions can improve the approximation quality.

References

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