Randomized local model order reduction for nonlinear PDEs

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We present a localized model order reduction (LMOR) approach for nonlinear (elliptic) partial differential equations (PDEs). LMOR methods are designed to deal with large-scale problems for which full order solves are not affordable in a reasonable time frame and problems with (local) topology changes that prevent the application of monolithic model order reduction techniques.
For linear problems optimal local approximation spaces in the sense of Kolmogorov are spanned by the left singular vectors of a transfer operator [1, 2]. The latter maps unknown Dirichlet boundary data on the boundary of an oversampling domain to the corresponding solution of the local PDE restricted to the subdomain or interface for which one wishes to generate a local reduced space; the boundary of the oversampling domain has to have a certain distance to the target subdomain or interface.
For nonlinear problems we again consider a transfer operator that maps unknown boundary data on the boundary of the oversampling domain to the corresponding local solution of the nonlinear PDE restricted to the target subdomain. Thanks to Caccioppoli's inequality, at the core of the analysis for linear problems, the range of this transfer operator and thus the (nonlinear) set of local solutions of the PDE on the target subdomain is compact. We then use the proper orthogonal decomposition (POD) to optimally approximate this compact set of solutions to a PDE dependent on parameters; the latter is here the unknown boundary data on the boundary of the oversampling domain. However, due to the high-dimensional parameter space, the POD suffers from the curse of dimensionality. To break the curse of dimensionality, we propose a randomized POD [3]. In detail, we consider random boundary conditions of controlled smoothness on the boundary of the oversampling domain, therefore introducing a probability distribution in parameter space. We also derive probabilistic a priori and a posteriori error bounds for the approximation error. Numerical experiments for a nonlinear diffusion problem demonstrate that already a local reduced space of low dimension yields a very accurate approximation.

References

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