On Balanced Truncation Error Bound and Sign Parameters

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Balanced truncation model reduction [2, 3] is a powerful tool for reducing linear time-invariant (LTI) dynamical systems, yielding reduced-order models that satisfy a simple *a priori* error bound in terms of the system's Hankel singular values. Consider the single-input, single-output (SISO) system

$$\mathcal{G} : \left\{ \quad x'(t) = Ax(t) + bu(t), \qquad y(t) = cx(t), \right. \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$, and $c \in \mathbb{R}^{1\times n}$. We assume (1) is asymptotically stable and minimal. Such a system is called *principal-axis balanced* if the unique solutions $\mathcal{P}$ and $\mathcal{Q}$ to the Lyapunov equations

$$A\mathcal{P} + \mathcal{P}A^\top + bb^\top = 0 \quad \text{and} \quad A^\top\mathcal{Q} + \mathcal{Q}A + c^\top c = 0$$

satisfy $\mathcal{P} = \mathcal{Q} = \Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ are the *Hankel singular values* of $\mathcal{G}$. In this balanced realization, $\mathcal{G}$ satisfies the sign-symmetry condition

$$A = SA^\top S, \quad b = (cS)^\top,$$

where $S = \operatorname{diag}(s_1, \ldots, s_n)$ and $\{s_i\}_{i=1}^n \subset \{\pm 1\}$ are the *sign parameters* corresponding to the Hankel singular values. Balanced truncation reduces the model order by removing components of the state space that correspond to small Hankel singular values, resulting in a reduced-order model $\mathcal{G}_r$ satisfying the $\mathcal{H}_\infty$ error bound [1]:

$$\|\mathcal{G} - \mathcal{G}_r\|_{\mathcal{H}_\infty} \leq 2(\sigma_{r+1} + \cdots + \sigma_n), \tag{2}$$

assuming the neglected Hankel singular values are distinct. First, we show that the balanced truncation error bound (2) holds with equality for SISO systems when sign parameters corresponding to the *truncated* Hankel singular values are consistent, that is $s_i = +1$ or -1 for $i = r+1, \ldots, n$. Second, we show how to determine these sign parameters for systems having *arrowhead realizations* of the form

$$A = \begin{bmatrix} d_1 & \alpha_2 & \cdots & \alpha_n \\ \beta_2 & d_2 & & \\ \vdots & & \ddots & \\ \beta_n & & & d_n \end{bmatrix}, \quad b = \gamma e_1, \quad \text{and} \quad c = e_1^\top,$$

where $\gamma \in \mathbb{R}$ and $e_1 \in \mathbb{R}^n$ is the first canonical basis vector. We prove that the sign parameters s_i are determined, up to a permutation, by the signs of the off-diagonal entries of the corresponding arrowhead realization: Namely, there exists a permutation $\{\pi_1, \pi_2, \ldots, \pi_n\}$ of $\{1, 2, \ldots, n\}$ such that

$$s_{\pi_1} = \operatorname{sign}(\gamma), \quad s_{\pi_i} = \operatorname{sign}(\gamma\alpha_i\beta_i), \quad i = 2, \ldots, n.$$

We illustrate these results with an arrowhead system arising in power systems modeling.

References

[1] D. F. Enns. Model reduction with balanced realizations: An error bound and a frequency weighted generalization. In *The 23rd IEEE Conference on Decision and Control*, pages 127–132. IEEE, 1984.

[2] B. Moore. Principal component analysis in linear systems: Controllability, observability, and model reduction. *IEEE Trans. Auto. Control*, 26(1):17–32, 1981.

[3] C. Mullis and R. Roberts. Synthesis of minimum roundoff noise fixed point digital filters. *IEEE Trans. Circuits Syst.*, 23(9):551–562, 1976.